ON THE INFLUENCE OF DISSIPATIVE GYROSCOPIC FORCES ON THE CONTROL LIABILITY AND OBSERVABILITY OF MECHANICAL SYSTEMS

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Necessary and sufficient conditions are established under which an incompletely controllable, conservative, mechanical system can be made completely controllable, stabilizable and observable in the neighborhood of steady-state motion by the application of dissipative and gyroscopic forces. Note that paper [1] has considered the influence of dissipative and gyroscopic forces on the controllability properties of conservative mechanical systems in certain special cases.

1. Let us consider a holonomic, conservative, mechanical system controlled by one controlling action. It is well-known [2] that in the neighborhood of equilibrium the linear approximation of such a system can be represented in the form

$$y_i'' = \lambda_i y_i + \alpha_i u \qquad (i = 1, \ldots, n) \tag{1.1}$$

It is also known [3] that system (1.1) is completely controllable by the action u if and only if

$$\lambda_i \neq \lambda_j, \quad \alpha_i \neq 0, \quad (i, j = 1, \ldots, n, \ i \neq j) \tag{1.2}$$

Let us assume that system (1.1) is not completely controllable by the action u. This signifies that there is equality among the λ_i or that some of the numbers α_i are equal to zero.

Below we investigate the following question: do there exist dissipative forces such that the system (1.1), incompletely controllable by action u, becomes completely controllable by this action in the presence of dissipation? The necessary conditions for the solvability of this problem can be formulated by the following theorems.

T h e o r e m 1.1 . If among the numbers λ_i at least two are equal to zero, or if $\lambda_i = \alpha_i = 0$ in even one of the equations of system (1.1), then system (1.1) cannot be completely controllable by action u no matter what dissipative or gyroscopic forces are supplementarily applied in this system.

Indeed, in the cases indicated at least one of the equations in (1.1) can be brought to the form $y_i'' = 0$ by means of a nonsingular linear transformation.

Let some dissipative or gyroscopic forces act supplementarily on the system. Then, Equation $y_i'' = 0$ takes the form

$$y''_i = a_1 y_1' + \ldots + a_n y_n' \tag{1.3}$$

and, consequently, the quantity $y_i' - a_1 y_1 - \ldots - a_n y_n = \text{const}$ is the first integral of system (1.1) in the presence of the dissipative and gyroscopic forces independently of u. Obviously, a system is not completely controllable if it admits of even one first integral which is independent of the controlling action. This proves the assertion.

An analogous assertion holds for the nonlinear case if there exist more than two cyclic coordinates or if for one coordinate the corresponding value $\alpha = 0$. Consider the system

$$y_i'' = \lambda_i y_i + \alpha_i u \quad (i = 1, \ldots, k), \qquad y_i'' = \lambda_i y_i \quad (i = k+1, \ldots, n) \tag{1.4}$$

Here

$$\begin{gathered} \lambda_i \neq \lambda_j, \quad \alpha_i \neq 0 \quad (i, j = 1, \ldots, k,\ i \neq j) \\ \lambda_i \neq 0 \quad (i = 1, \ldots, k-1,\ k+1, \ldots, n) \end{gathered} \tag{1.5}$$

Let us assume that besides the conservative forces and the controlling action, dissipative forces act on system (1.4).

T h e o r e m 1.2. The fulfillment of conditions (1.5) is sufficient for the existence of dissipative forces such that the incompletely controllable mechanical system (1.1) becomes completely controllable in the presence of dissipation.

P r o o f . Let the dissipative forces be generated by the Rayleigh function

$$2R = \sum_{i=1}^{k-1} \gamma_{ii} y_i'^2 + \sum_{i=k}^{n} (\gamma_{ii} y_i'^2 + 2\gamma_{ii+1} y_i' y_{i+1}') \tag{1.6}$$

Here R is positive definite; $\gamma_{n\,n+1} = 0$, i.e. we consider the system

$$\begin{gathered} x_{2i-1}' = x_{2i}, \quad x_{2i}' = \lambda_i x_{2i-1} - \frac{\partial R}{\partial x_{2i}} + \alpha_i u \quad (i = 1, \ldots, k) \\ x_{2i-1}' = x_{2i}, \quad x_{2i}' = \lambda_i x_{2i-1} - \frac{\partial R}{\partial x_{2i}} \quad (i = k+1, \ldots, n) \end{gathered} \tag{1.7}$$

which in vector-matrix form will be $x' = Ax + bu$.

For an affirmative answer to the question we have posed it suffices to show that under a suitable choice of numbers γ_{ij} satisfying the condition that the function R in (1.6) be sign-positive:

1) the eigenvalues of matrix A will be real and distinct;
2) the projection of vector b on any row of the matrix S^{-1} (where S is the fundamental matrix of matrix A) differes from zero.

For system (1.7) we construct the characteristic equation $|A - \mu E| = 0$ which in expanded form will be

$$(\lambda_1 - \gamma_{11}\mu - \mu^2) \ldots (\lambda_{k-1} - \mu\gamma_{k-1,\,k-1} - \mu^2)\, \Delta_{2p}(\mu) = 0 \qquad (p = n - k + 1) \tag{1.8}$$

Here $\Delta_{2p}(\mu)$ is determined from the recurrence realizations

$$\Delta_{2i}(\mu)=(-\mu^2-\gamma_{n-i+1,\,n-i+1}\mu+\lambda_{n-i+1})\,\Delta_{2i-2}(\mu)-\mu^2\gamma^2_{n-i+1,\,n-i+2}\,\Delta_{2i-4}(\mu)$$
$$(i=1,\ldots,p,\ \Delta_0=1,\ \Delta_{-2}=0) \tag{1.9}$$

Let us show that under a proper choice of γ_{ij}, Equation $\Delta_{2i}(\mu)=0$ has $2i$ distinct real roots. We denote them by $\mu_1^{(2i)},\ldots,\mu_{2i}^{(2i)}$. The proof will be carried out by induction.

From (1.9), when $\gamma_{nn}^2>-4\lambda_n$, $\Delta_2(\mu)=\mu^2+\gamma_{nn}\mu-\lambda_n=0$ has the two real roots

$$\mu_{1,2}^{(2)}=-{}^1\!/_2\gamma_{nn}\pm\sqrt{{}^1\!/_4\gamma_{nn}^2+\lambda_n}$$

which also are distinct.

We show that under a proper choice of the numbers $\gamma_{n-1,\,n-1}, \gamma_{n-1,\,n}$ Equation $\Delta_4(\mu)=0$ has four real roots. Since $\Delta_4(\mu)\to+\infty$ as $\mu\to-\infty$ and since it follows from (1.9) that $\Delta_4(\mu_1^{(2)})<0$, being a continuous function, $\Delta_4(\mu)$ has at least one real root in the interval $-\infty<\mu<\mu_1^{(2)}$. In just the same way it can be verified that the function $\Delta_4(\mu)$ has at least one more real root in the interval $\mu_2^{(2)}<\mu<+\infty$. Let $\varepsilon>0$ be an arbitrary number. Under the conditions

$$\gamma_{ii}>\varepsilon-\varepsilon^{-1}\lambda_i,\qquad \gamma_{ii}^2+4\lambda_i>0\qquad (i=k,\ldots,n) \tag{1.10}$$

imposed on γ_{ii}, there holds the relation

$$-(-\varepsilon)^2-\gamma_{ii}(-\varepsilon)+\lambda_i>0 \tag{1.11}$$

i.e. the point $\mu=-\varepsilon$ is to be found between the roots of Equation

$$-\mu^2-\gamma_{ii}\mu+\lambda_i=0\qquad (i=k,\ldots,n) \tag{1.12}$$

We chose $\gamma_{n-1,n}$ such that

$$\Delta_4(-\varepsilon)=(-\varepsilon^2+\gamma_{n-1,\,n-1}\varepsilon+\lambda_{n-1})\,\Delta_2(-\varepsilon)-\varepsilon^2\gamma^2_{n,\,n-1}>0$$

for which it is sufficient to require that

$$\gamma^2_{n-1,\,n}<\varepsilon^{-2}\left[(-\varepsilon^2+\gamma_{n-1,\,n-1}\varepsilon+\lambda_{n-1})\,\Delta_2(-\varepsilon)\right] \tag{1.13}$$

From (1.10) and (1.13) it follows that the indicated choices of γ_{nn}, $\gamma_{n-1,n-1}$ and $\gamma_{n-1,n}$ do not contradict the positive definiteness of R in (1.6). On the other hand, under the indicated choices of γ_{nn}, $\gamma_{n-1,n-1}$ and $\gamma_{n,n-1}$ the function $\Delta_4(\mu)$, being a continuous function in the intervals $(-\varepsilon,\mu_2^{(2)})$ and $(\mu_1^2-\varepsilon)$, has at least one root in each.

But since the number of roots of $\Delta_4(\mu)$ cannot be larger than four, then under the stated choices of γ_{ij}, $\Delta_4(\mu)$ has four real, distinct roots distributed in the following order

$$\mu_1^{(4)}<\mu_1^{(2)}<\mu_2^{(4)}<-\varepsilon<\mu_3^{(4)}<\mu_2^{(2)}<\mu_4^{(4)}$$

Let us assume that the numbers $\gamma_{n-j+1,\,n-j+1},\gamma_{n-j+1,\,n-j+2}$ $(j=1,\ldots,i-1)$ are chosen such that the roots of $\Delta_{2i-2}(\mu)$ and $\Delta_{2i-4}(\mu)$ are real, distinct and distributed as follows:

$$\mu_1^{(2i-2)}<\mu_1^{(2i-4)}<\mu_2^{(2i-2)}<\mu_2^{(2i-4)}<\cdots<\mu_{i-2}^{(2i-2)}<\mu_{i-2}^{(2i-4)}<\mu_{i-1}^{(2i-2)}<-\varepsilon<\mu_i^{(2i-2)}<$$
$$<\mu_{i-1}^{(2i-4)}<\mu_{i+1}^{(2i-2)}<\cdots<\mu_{2i-5}^{(2i-4)}<\mu_{2i-3}^{(2i-2)}<\mu_{2i-4}^{(2i-4)}<\mu_{2i-2}^{(2i-2)} \tag{1.14}$$

and let us show that under (1.10) the numbers $\gamma_{n-i+1,\,n-i+2}$ can be chosen such that the roots of $\Delta_{2i}(\mu)$ and $\Delta_{2i-2}(\mu)$ would be distributed analogously to (1.14). From (1.9), (1.11), (1.14) and from the fact that $\Delta_{2i}(\mu)\to(-1)^i\infty$ as $\mu\to\pm\infty$ $(i=1,\ldots,p)$, it follows that

$$\operatorname{sign}\Delta_{2i}(\mu_1^{(2i-2)})=(-1)^{i-1},\quad \operatorname{sign}\Delta_{2i}(\mu_{2i-2}^{(2i-2)})=(-1)^{i-1}$$
$$\operatorname{sign}\Delta_{2i}(\mu_2^{(2i-2)})=(-1)^{i-2},\quad \operatorname{sign}\Delta_{2i}(\mu_{2i-3}^{(2i-2)})=(-1)^{i-2} \tag{1.15}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\Delta_{2i}(\mu_{i-1}^{(2i-2)})<0,\quad \Delta_{2i}(\mu_i^{(2i-2)})<0$$

then $\Delta_{2i}(\mu)$, being a continuous function, has at least one root in each of the intervals

$$(-\infty, \mu_1^{(2i-2)}),\ (\mu_1^{(2i-2)}, \mu_2^{(2i-2)}), \ldots, (\mu_{i-2}^{(2i-2)}, \mu_{i-1}^{(2i-2)})$$
$$(\mu_i^{(2i-2)}, \mu_{i+1}^{(2i-2)}), \ldots, (\mu_{2i-3}^{(2i-2)}, \mu_{2i-2}^{(2i-2)}),\ (\mu_{2i-2}^{(2i-2)}, +\infty) \tag{1.16}$$

But since

$$\Delta_{2i}(\mu_{i-1}^{(2i-2)}) < 0, \qquad \Delta_{2i}(\mu_i^{(2i-2)}) < 0, \qquad \Delta_{2i-2}(-\varepsilon) > 0, \qquad \Delta_{2i-1}(-\varepsilon) > 0$$

then from (1.9)

$$(-\varepsilon^2 + \gamma_{n-i+1,\, n-i+1}\,\varepsilon + \lambda_{n-i+1})\,\Delta_{2i-2}(-\varepsilon) > 0$$

We choose $\gamma_{n-i+1,\, n-i+2}$ such that

$$\gamma_{n-i+1,\, n-i+2}^2 < \frac{(-\varepsilon^2 + \gamma_{n-i+1,\, n-i+1}\,\varepsilon + \lambda_{n-i+1})\,\Delta_{2i-2}(-\varepsilon)}{\varepsilon^2 \Delta_{2i-1}(-\varepsilon)} \tag{1.17}$$

Then in accordance with (1.9) we have $\Delta_{2i}(-\varepsilon) > 0$. This signifies that $\Delta_{2i}(\mu)$ changes sign in the intervals $(\mu_i^{(2i-2)}, -\varepsilon)$ and $(-\varepsilon, \mu_i^{(2i-2)})$ and, consequently, $\Delta_{2i}(\mu)$ has one root in each of these intervals since $\Delta_{2i}(\mu)$ cannot have more than $2i$ roots.

The roots of $\Delta_{2i}(\mu)$ and $\Delta_{2i-2}(\mu)$ can not coincide since otherwise it would follow from Formula (1.9) that $\Delta_0 = 0$, but, $\Delta_0 = 1$. Thus we can assert that under a suitable choice of $\gamma_{ii}, \gamma_{i-1,i}$ $(i = k, \ldots, n)$, $\Delta_{2p}(\mu)$ has $2p$ real, distinct roots which do not coincide with the roots of $\Delta_{2p-2}(\mu)$. Among the roots $\mu_j^{(2p)}$ $(j = 1, \ldots, 2p)$ only one may be zero and, moreover, if and only if $\lambda_k = 0$. Let us assume after this that only $\mu_1^{(2p)}$ can be zero while the remaining roots are nonzero. From (1.9) it follows also that the zero root $\mu_1^{(2p)}$ does not depend on γ_{ij} while the remaining roots are nonzero for any γ_{ij}.

It follows from Equation (1.8) that under the conditions $\gamma_{ii}^2 > -4\lambda_i$ $(i = 1, \ldots, k-1)$ the remaining $2k - 2$ roots of this equation are real and nonzero as well.

Let

$$\tfrac{1}{2}\,|-\gamma_{ii} + \sqrt{\gamma_{ii}^2 + 4\lambda_i}\,| < \min_m |\mu_m^{(2p)}|$$
$$\tfrac{1}{2}\,|\gamma_{ii} + \sqrt{\gamma_{ii}^2 + 4\lambda_i}\,| < \max_m |\mu_m^{(2p)}|,\ \mu_m^{(2p)} \neq 0 \tag{1.18}$$
$$-\gamma_{ii} \pm \sqrt{\gamma_{ii}^2 + 4\lambda_i} \neq -\gamma_{ll} \pm \sqrt{\gamma_{ll}^2 + 4\lambda_l}$$
$$(i, l = 1, \ldots, k-1, \quad i \neq l, \quad m = 1, 2, \ldots, 2p,\ m \neq j \ \text{ for } \ \mu_j^{(2p)} = 0)$$

This can be achieved by increasing the γ_{ii} $(i = 1, \ldots, k-1)$, without violating the positive-definiteness of form R. Then, all the roots of Equation (1.8) will be real and distinct; we denote these roots by μ_j $(j = 1, \ldots, 2n)$ and only μ_{2k-1} will be zero.

To prove the second part of the theorem we must show that not one of the rows of the matrix S^{-1} is perpendicular to the $2n$-dimensional column-vector

$$b = \{0, \alpha_1, 0, \ldots, 0, \alpha_k, 0, \ldots, 0\} \tag{1.19}$$

where S is the fundamental matrix of matrix A:

$$S = \left\| \begin{matrix} 1 & 1 & \ldots 0 & 0 & 0 & \ldots & 0 \\ \mu_1 & \mu_2 & \ldots 0 & 0 & 0 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots 1 & 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots \mu_{2k-3} & \mu_{2k-2} & 0 & \ldots & 0 \\ 0 & 0 & \ldots 0 & 0 & s_{2k-1}^{(2k-1)} & \ldots & s_{2k-1}^{(2n)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots 0 & 0 & s_{2n}^{(2k-1)} & \ldots & s_{2n}^{(2n)} \end{matrix} \right\| \tag{1.20}$$

The numbers $s_i^{(j)}$ $(i = 2k-1, \ldots, 2n)$, the components of the jth eigenvector of matrix A $(j = 2k-1, \ldots, 2n)$, satisfy the system of equations

$$s_{2k}^{(j)} = \mu_j s_{2k-1}^{(j)}$$

$$\lambda_k s_{2k-1}^{(j)} - \gamma_{kk} s_{2k}^{(j)} - \gamma_{kk+1} s_{2k+2}^{(j)} = \mu_j s_{2k}^{(j)}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-\gamma_{n-1,n} s_{2n-2}^{(j)} + \lambda_n s_{2n-1}^{(j)} - \gamma_{nn} s_{2n}^{(j)} = \mu_j s_{2n}^{(j)} \qquad (j = 2k-1, \ldots, 2n)$$

This system can be put in the form (1.21)

$$\gamma_{n-i,\, n-i+1} \mu_j s_{2n-2i-1}^{(j)} = (\lambda_{n-i+1} - \mu_j \gamma_{n-i+1,\, n-i+1} - \mu_j^2) s_{2n-2i+1}^{(j)} - \mu_j \gamma_{n-i+1,\, n-i+2} s_{2n-2i+3}^{(j)}$$

$$s_{2n-2i+2}^{(j)} = \mu_j s_{2n-2i+1}^{(j)}, \qquad s_{2n-1}^{(j)} = 1, \qquad s_{2n+1}^{(j)} = 0, \qquad \gamma_{k-1k} = \gamma_{nn+1} = 0$$

$$(i = 1, \ldots, n-k+1,\ j = 2k-1, \ldots, 2n)$$

Let us replace the $s_i^{(j)}$ in accordance with Formula

$$s_{2n-2i+1}^{(j)} = \gamma_{n-1n}^{-1} \gamma_{n-2,\, n-1}^{-1} \cdots \gamma_{n-i+1,\, n-i+2}^{-1} v_{2n-2i+1}^{(j)}$$

$$(i = 1, \ldots, n-k+1,\ j = 2k-1, \ldots, 2n,\ v_{2n-1}^{(j)} = s_{2n-1}^{(j)} = 1)$$

Then Equations (1.21) take the form

$$\mu_j v_{2n-2i-1}^{(j)} = v_{2n-2i+1}^{(j)} (\lambda_{n-i+1} - \mu_j \gamma_{n-i+1,\, n-i+1} - \mu_j^2) - \mu_j \gamma_{n-i+1,\, n-i+2}^2 v_{2n-2i+3}^{(j)}$$

$$(i = 1, \ldots, n-k+1,\ j = 2k-1, \ldots, 2n,\ v_{2n+1}^{(j)} = 0) \qquad (1.22)$$

In case $\mu_j \neq 0$ $(j = 2k, \ldots, 2n)$ when $\lambda_k = 0$ the components of the $(2k-1)$-st eigenvector of matrix A take the form

$$s_1^{(2k-1)} = 0, \ldots, s_{2k-2}^{(2k-1)} = 0, \qquad s_{2k-1}^{(2k-1)} = 1, \qquad s_{2k}^{(2k-1)} = 0, \ldots, s_{2n}^{(2k-1)} = 0 \qquad (1.23)$$

Equation (1.22) here becomes an identity. Therefore, dividing Equation (1.22) by μ_j $(j = 2k, \ldots, 2n)$, we get

$$v_{2n-2i-1}^{(j)} = v_{2n-2i+1}^{(j)} \left(\frac{\lambda_{n-i+1}}{\mu_j} - \gamma_{n-i+1,\, n-i+1} - \mu_j \right) - \gamma_{n-i+1,\, n-i+2}^2 v_{2n-2i+3}^{(j)} \qquad (1.24)$$

Note that not one of Equations $\Delta_{2i}(\mu) = 0$ $(i = 1, \ldots, p-1)$ in (1.9) can have a zero root and, therefore, when constructing system (1.9) we could take

$$D_{2i}(\mu) = \left(\frac{\lambda_{n-i+1}}{\mu} - \mu - \gamma_{n-i+1,\, n-i+1} \right) D_{2i-2}(\mu) - \gamma_{n-i+1,\, n-i+2}^2 D_{2i-4}(\mu)$$

$$(i = 1, \ldots, p-1,\ D_0 = 1) \qquad (1.25)$$

as the recurrence relations.

It is obvious that the roots $D_{2i}(\mu)$ and $\Delta_{2i}(\mu)$ $(i = 1, \ldots, p-1)$ coincide. From Equations (1.24) and (1.25) it follows that

$$v_{2n-2i+1}^{(j)} = D_{2i}(\mu_j) \qquad (i = 0, \ldots, n-k,\ j = 2k, \ldots, 2n) \qquad (1.26)$$

Consequently,

$$s_{2n-2i+1}^{(j)} = \gamma_{n-1,\, n}^{-1} \gamma_{n-2,\, n-1}^{-1} \cdots \gamma_{n-i+1,\, n-i+2}^{-1} D_{2i-2}(\mu_j), \qquad s_{2n-2i+2}^{(j)} = \mu_j s_{2n-2i+1}^{(j)} \qquad (1.27)$$

We need to compute the determinant of the matrix S in order to find the matrix S^{-1}. Since the matrix S has a quasidiagonal structure, then

$$|S| = \sqrt{\gamma_{11}^2 + 4\lambda_1} \cdots \sqrt{\gamma_{k-1,\, k-1}^2 + 4\lambda_{k-1}} \begin{vmatrix} s_{2k-1}^{(2k-1)} & \cdots & s_{2k-1}^{(2n)} \\ \cdots & \cdots & \cdots \\ s_{2n}^{(2k-1)} & \cdots & s_{2n}^{(2n)} \end{vmatrix}$$

Multiplying every column of the determinant by μ_j^{n-k} $(j = 2k-1, \ldots, 2n,$ $\mu_{2k-1} \neq 0)$, we reduce it to the Vandermonde determinant, after which we get

$$|S| = (-1)^{\alpha}\sqrt{\gamma_{11}^2 + 4\lambda_1}\cdots\sqrt{\gamma^2_{k-1,\,k-1} + 4\lambda_{k-1}}\ \frac{\gamma_{n-1n}^{2k-2n}\cdots\gamma_{kk+1}^{-2}\lambda_n^{n-k}\cdots\lambda_{k+1}}{\mu_{2k-1}^{n-k}\cdots\mu_{2n}^{n-k}}\times$$

$$\times(\mu_{2k} - \mu_{2k-1})\cdots(\mu_{2n} - \mu_{2n-1}) \quad \text{for } \mu_{2k-1} \neq 0 \tag{1.28}$$

However, when $\lambda_k = 0,\ \mu_{2k-1} = 0,$ then to compute $|S|$ it suffices to divide and multiply by $\mu_{2k}^{n-k}\ldots\mu_{2n}^{n-k}$ and we get a formula similar to (1.28). Let us denote the elements of the matrix S^{-1} by p_{ij}. From (1.20) it follows that

$$S^{-1} = \left\|\begin{matrix} p_{11} & p_{12}\ldots 0 & 0 & 0 & \ldots 0 \\ p_{21} & p_{22}\ldots 0 & 0 & 0 & \ldots 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0\ \ldots\ p_{2k-2,\,2k-3} & p_{2k-3,\,2k-2} & 0 & \ldots 0 \\ 0 & 0\ \ldots\ p_{2k-2,\,2k-3} & p_{2k-2,\,2k-2} & 0 & \ldots 0 \\ 0 & 0\ \ldots 0 & 0 & p_{2k-1,\,2k-1} & \ldots p_{2k-1,\,2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0\ \ldots 0 & 0 & p_{2n,\,2k-1} & \ldots p_{2n,\,2n} \end{matrix}\right\| \tag{1.29}$$

To prove that not one of the rows of matrix S^{-1} is perpendicular to the vector b in (1.19) it is sufficient to show that

$$p_{12} = -\frac{1}{\sqrt{\gamma_{11}^2 + 4\lambda_1}} \neq 0, \qquad p_{22} = \frac{1}{\sqrt{\gamma_{22}^2 + 4\lambda_2}} \neq 0, \ldots$$

$$p_{2k-3,\,2k-2} = -\frac{1}{\sqrt{\gamma^2_{k-1,\,k-1} + 4\lambda_{k-1}}} \neq 0, \qquad p_{2k-2,\,2k-2} = \frac{1}{\sqrt{\gamma^2_{k-1,\,k-1} + 4\lambda_{k-1}}} \neq 0$$

$$p_{2k-1,\,2k} = (-1)^{\beta_1}\frac{\gamma_{n-1,n}\cdots\gamma_{k,k+1}\mu_{2k-1}^{n-k}}{(\mu_{2k} - \mu_{2k-1})\cdots(\mu_{2n} - \mu_{2n-1})} \neq 0 \quad \text{for } \mu_{2k-1} \neq 0 \tag{1.30}$$

$$p_{2k-1,\,2k} = (-1)^{\beta_2}\frac{\gamma_{n-1,n}\cdots\gamma_{k,k+1}}{\mu_{2k}\cdots\mu_{2n}} \neq 0 \quad \text{for } \mu_{2k-1} = 0$$

$$p_{j2k} = (-1)^{\beta_3}\frac{\gamma_{n-1,n}\cdots\gamma_{k,k+1}\mu_j^{n-k}}{(\mu_{2k-1} - \mu_j)\ldots(\mu_{j-1} - \mu_j)(\mu_j - \mu_{j+1})\cdots(\mu_j - \mu_{2n})} \neq 0$$

$$(j = 2k, \ldots, 2n,\ \beta_1, \beta_2, \beta_3 - \text{integers})$$

And this follows immediately from the course of the proof of Theorem 1.2.

Thus, when conditions (1.5) are satisfied, the dissipative forces can be chosen in accordance with (1.6),(1.10),(1.17),(1.18). According to the theorem on the duality [4] between complete controllability and observability, system (1.1) which is incompletely observable with respect to the quantity $\xi = (cx)$ (where $c = \{c_1, 0, \ldots, c_k, 0, \ldots, 0\}$) can be made completely observable if, in addition to the conservative forces, dissipative forces are applied to the system in the above-mentioned manner. If we further assume that $\lambda_k \neq 0$, then under (1.5) system (1.1) is incompletely observable with respect to the rate $\xi' = (b^*x)$, but in the presence of dissipative forces in the manner stated above, the system can be made completely observable with respect to the quantity $\xi' = (b^*x)$.

In both cases conditions (1.5) are necessary and sufficient for the existence of the dissipative forces which make system (1.1) completely observable with respect to the quantities $\xi = (cx)$ and $\xi' = (b^*x)$.

2. In order to study the improvement in the controllability of system (1.1) due to the application of gyroscopic forces, we can prove the sufficiency of the conditions of Theorem 1.1 in the presence of which system (1.4) becomes completely controllable.

P r o o f . Let gyroscopic forces be applied such that in their presence the system (1.4) takes the following form:

$$x'_{2j-1} = x_{2j}, \quad x_{2i}' = \lambda_i x_{2i-1} + \alpha_i u \quad (i = 1, \ldots, k-1)$$
$$x_{2k}' = \lambda_k x_{2k-1} + \omega_k x_{2k+2} + \alpha_k u \tag{2.1}$$
$$x_{2i}' = -\omega_{i-1} x_{2i-2} + \lambda_i x_{2i-1} + \omega_i x_{2i+2} \quad (i = k+1, \ldots, n, \ j = 1, \ldots, n, \ w_n = 0)$$

and, in vector-matrix form, $x' = Ax + bu$.

Let us reduce system (2.1) to the first normal form ([5], p.125) by means of nonsingular, real, linear transformation. For this we must determine the elementary divisors of the matrix $A - \mu E$. We formulate Equation

$$|A - \mu E| = 0 \tag{2.2}$$

which in expanded form will be

$$(\mu^2 - \lambda_1)(\mu^2 - \lambda_2) \ldots (\mu^2 - \lambda_{k-1}) \Delta_{2p}(\mu) = 0 \qquad (p = n - k + 1)$$

Here $\Delta_{2p}(\mu)$ is determined from the recurrence relations

$$\Delta_{2i}(\mu) = (\mu^2 - \lambda_{n-i+1}) \Delta_{2i-2}(\mu) + \omega^2_{n-i+1}\mu^2 \Delta_{2i-4}(\mu) \quad (i = 1, \ldots, p; \ \Delta_0 = 1; \ \Delta_{-2} = 0) \tag{2.3}$$

Denoting $\mu^2 = \nu$, let us show that the numbers ω_j can be chosen such that Equation $\Delta_{2i}(\nu) = 0$ would have i distinct, real roots not coincident with the roots of $\Delta_{2i-2}(\nu) = 0$. We prove this by induction. When $i = 1$ this is obvious since λ_n is a real number while $\Delta_2(\nu) = \nu - \lambda_n = 0$ or $\nu_1^{(2)} = \lambda_n$. When $i = 2$ we have

$$\Delta_4(\nu) = (\nu - \lambda_{n-1}) \Delta_2(\nu) + \omega^2_{n-1}\Delta_0$$

Let $-\varepsilon < 0$ be an arbitrary number not coinciding with the numbers $\lambda_{n-1}, \nu^{(2)}$, i.e. with the roots of $(\lambda_{n-1} - \nu)\Delta_2(\nu) = 0$, Then, when

$$\omega^2_{n-1} > \frac{-(\lambda_{n-1} + \varepsilon)\Delta_2(-\varepsilon)}{\varepsilon \Delta_0}, \qquad \Delta_4(-\varepsilon) < 0$$

But since $\Delta_4(\nu) \to +\infty$ as $\nu \to \pm\infty$, then $\Delta_4(\nu)$ has the real, distinct roots $\nu_1^{(4)}, \nu_2^{(4)}$. The roots of $\Delta_4(\nu)$ and $\Delta_2(\nu)$ cannot coincide since otherwise $\Delta_0 = 0$. Thus we have proved that $\Delta_4(\nu)$ has two real, distinct roots not coinciding with $\nu_1^{(2)}$.

Let us assume that the functions $\Delta_{2i-2}(\nu)$ and $\Delta_{2i-4}(\nu)$ have, respectively, $i - 1$ and $i - 2$ real, distinct roots which do not coincide.

We show that ω_{n-i+1} can be secected such that $\Delta_{2i}(\nu)$ has i real, distinct roots not coincident with the roots of $\Delta_{2i-2}(\nu)$. We write out $\Delta_{2i}(\nu)$ as follows:

$$\Delta_{2i}(\nu) = (\nu - \lambda_{n+1-i}) \Delta_{2i-2}(\nu) + \omega_{n-i+1}\nu\Delta_{2i-4}(\nu) \tag{2.4}$$

Since $\Delta_{2j}(\nu)$ is a polynomial of order j in ν and the coefficient of its leading term is unity $(j = 0, \ldots, p)$, then

$$\operatorname{sign} \Delta_{2i}(-\infty) = (-1)^i, \qquad \operatorname{sign} \Delta_{2i-4}(-\infty) = (-1)^i \tag{2.5}$$
$$\Delta_{2i}(+\infty) > 0, \quad \Delta_{2i-4}(+\infty) > 0$$

Let

$$\varepsilon > \max_j |\nu_j^{(2i-4)}| \quad (1 \leqslant j \leqslant i-2)$$

Here $\nu_j^{(2i-4)}$ are the roots of Equation $\Delta_{2i-4}(\nu) = 0$. Then $\operatorname{sign} \Delta_{2i}(-\varepsilon) = (-1)^{i-1}$ under (2.4),(2.5) and

$$\omega^2_{n-i+1} > \frac{-(\varepsilon + \lambda_{n-i+1})\,\Delta_{2i-2}(\nu)}{\varepsilon\Delta_{2i-4}(-\varepsilon)}$$

Thus Equation $\Delta_{2i}(\nu) = 0$ has at least one root to the left of the point $\nu = -\epsilon$. Since $\nu = 0$ is not a root of $\Delta_{2i-4}(\nu)$, then $\nu\Delta_{2i-4}(\nu)$ has $i-1$ real, distinct roots, i.e. as ν is varied from $-\epsilon$ to $+\infty$, $\nu\Delta_{2i-4}(\nu)$ changes sign $i-1$ times. By choosing ω_{n-i+1} so large that the signs of $\nu\Delta_{2i-4}(\nu)$ and $\Delta_{2i}(\nu)$ coincide at least at one point in every interval included between the roots of $\nu\Delta_{2i-4}(\nu) = 0$, we obtain from (2.4) and (2.5) that $\Delta_{2i}(\nu)$ has $i-1$ changes of sign to the right of the point $\nu = -\epsilon$ and, consequently, $\Delta_{2i}(\nu)$, being a continuous function, has at least $i-1$ real roots to the right of the point $\nu = -\epsilon$.

From what we have said above it follows that by increasing ω_{n-i+1} we can make $\Delta_{2i}(\nu)$ have at least i real, distinct roots. But since $\Delta_{2i}(\nu)$ cannot have more than i roots, then $\Delta_{2i}(\nu)$ has precisely i real, distinct roots. The roots of $\Delta_{2i}(\nu)$ and $\Delta_{2i-2}(\nu)$ cannot coincide since otherwise either of the roots of $\Delta_{2i-4}(\nu)$ and $\Delta_{2i-2}(\nu)$ would coincide or $\nu = 0$ would be a root of $\Delta_{2i-2}(\nu)$.

But both these cases are impossible since in the first case, by induction, $\Delta_{2i-2}(\nu)$ and $\Delta_{2i-4}(\nu)$ cannot have common roots, while in the second, $\lambda_n \ldots \lambda_{n-i+1} = 0\ (i = 1, \ldots, p-1)$, which also is impossible according to (1.2)

Thus, we have proved that it is possible to select $\omega_k, \ldots, \omega_{n-1}$ such that the roots of $\Delta_{2i}(\nu)$ are real, distinct and noncoincident with the roots of $\Delta_{2i-2}(\nu)$, $(i = 1, \ldots, p)$. From conditions (1.2) it follows that only λ_k can be zero, and when $\lambda_k = 0$, (from (2.3)) $\Delta_{2p}(\nu)$ has one zero root independently of $\omega_k, \ldots, \omega_{n-1}$. Let us assume that when $\lambda_k = 0$ only $\nu_k^{(2p)}$ equals zero.

Since the roots of $\Delta_{2p-2}(\nu)$ and $\Delta_{2p-4}(\nu)$ do not coincide, it follows from (2.3) that λ_k is the only root that $\Delta_{2p}(\nu) = 0$ and $\Delta_{2p-4}(\nu) = 0$ can have in common. Consequently, in very small neighborhoods of the remaining $p-1$ roots of Equation $\Delta_{2p}(\nu) = 0$, the derivative

$$\frac{d\Delta_{2p}(\nu)}{d\omega_k^2} = \nu\Delta_{2p-4}(\nu) \neq 0$$

i.e. there exist monotonous functions $f_j(\omega_n^2) = \nu_j^{(2p)}\ (j = 2, \ldots, p)$, which express the roots $\nu_j^{(2p)}$ in terms of ω_k^2. Hence it follows that under small variations of ω_k^2 we can achieve that not one of the numbers $\lambda_1, \ldots, \lambda_{k-1}$ are coincident with the roots $\nu_2^{(2p)}, \ldots, \nu_p^{(2p)}$ of Equation $\Delta_{2p}(\nu) = 0$. If λ_k is a root of $\Delta_{2p-4}(\nu)$, then $\nu_1^{(2p)} = \lambda$ which also does not coincide with the numbers $\lambda_1, \ldots, \lambda_{k-1}$. However, if λ_k is not a root of Equation $\Delta_{2p-4}(\nu) = 0$, then $\nu_1^{(2p)} \neq \lambda_k$, and by means of varying ω_k^2 we can achieve that $\nu_1^{(2p)}$ also is not coincident with the numbers $\lambda_1, \ldots, \lambda_{k-1}$.

Thus, under (1.2) we can select the numbers $\omega_k, \ldots, \omega_{n-1}$ such that all the roots of Equation $(\nu - \lambda_1) \ldots (\nu - \lambda_{k-1})\,\Delta_{2p}(\nu) = 0$ are distinct and real. We denote by $\lambda_1, \ldots, \lambda_{k-1}, \nu_k, \ldots, \nu_n$, and here $\nu_k = 0$ when $\lambda_k = 0$. Thus, by a proper choice of $\omega_k, \ldots, \omega_{n-1}$, Equation (2.2) can be written as

$$(\mu^2 - \lambda_1) \ldots (\mu^2 - \lambda_{k-1})(\mu^2 - \nu_k) \ldots (\mu^2 - \nu_n) = 0 \tag{2.6}$$

Consequently ([5], p.127), by means of a nonsingular linear transformation (in the real number field) we can reduce matrix A to a quasidiagonal form each of whose diagonal cells is of second order and has the form

$$\begin{Vmatrix} 0 & 1 \\ \nu_i & 0 \end{Vmatrix} \tag{2.7}$$

Then in matrix symbolism system (2.1) takes the form

$$z' = SAS^{-1}z + Sbu, \quad (|S| \neq 0) \qquad \left(SAS^{-1} = \begin{Vmatrix} 0 & 1 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ \lambda_1 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & \lambda_{k-1} & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & \nu_k & 0 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & \nu_n & 0 \end{Vmatrix} = L\right) \tag{2.8}$$

We construct now the matrix equation $SA = LS$; it is equivalent to the system of equations

$$\begin{aligned} &\lambda_j s_{2i-1,2j} = s_{2i,2j}, \qquad s_{2i-1,2j-1} = s_{2i,2j} \\ &\lambda_j s_{2i,2j} = \lambda_i s_{2i-1,2j-1}, \qquad s_{2i-1,2j-1} = \lambda_i s_{2i-1,2j} \end{aligned} \qquad (i, j = 1, \ldots, k-1)$$

$$\begin{aligned} &\lambda_j s_{2i-1,2j} = s_{2i,2j}, \qquad \omega_{j-1} s_{2i-1,2j-2} + s_{2i-1,2j-1} - s_{2i-1,2j+2}\omega_j = s_{2i,2j} \\ &\lambda_j s_{2i,2j} = \nu_i s_{2i-1,2j-1}\omega_{j-1}, \qquad s_{2i,2j-2} + s_{2i,2j-1} - s_{2i\ 2j+2}\omega_j = \nu_i s_{2i-1,2j} \end{aligned} \qquad (i, j = k, \ldots, n) \tag{2.9}$$

To determine S it is sufficient to find such a solution of the homogeneous system (2.9) for which the determinant $|S| \neq 0$. Let us assume that

$$\begin{gathered} s_{ij} = \delta_{ij} \ (i, j = 1, \ldots, 2k-2, \ \delta_{ii} = 1, \ \delta_{ij} = 0, \ i \neq j) \\ s_{2i-1, 2j-\frac{1}{2}[1+(-1)^{j-k+1}]} = 0, \quad s_{2i, 2j-\frac{1}{2}[1-(-1)^{j-k+1}]} = 0 \ (i, j = k, \ldots, n) \end{gathered} \tag{2.10}$$

To determine the other elements of matrix S we consider two cases.

F i r s t c a s e . Let $p = n - k + 1$ be an even number; then we can satisfy system (2.9) if we select the remaining s_{ij} in the following manner:

$$\begin{gathered} s_{2i-1,2k+4j-5} = \lambda_{k+2j-2}\frac{\Delta_{2p-4j+2}(\nu_i)}{\omega_{n-1}\ldots\omega_{k+2j-2}}\nu_i^{j-2}, \qquad s_{2i-1,2k+4j-2} = \frac{\Delta_{2p-4j}(\nu_i)}{\omega_{n-1}\ldots\omega_{k+2j-1}}\nu_i^{j-1} \\ s_{2i,2k+4j-4} = \frac{\Delta_{2p-4j+2}(\nu_i)}{\omega_{n-1}\ldots\omega_{k+2j-2}}\nu_i^{j-1}, \qquad s_{2i,2k+4j-3} = \frac{\Delta_{2p-4j}(\nu_i)}{\omega_{n-1}\ldots\omega_{k+2j-1}}\lambda_{k+2j-1}\nu_i^{j-1} \\ (j = 1, \ldots, {}^1\!/_2 p = {}^1\!/_2[n-k+1], \quad i = k, \ldots, n) \end{gathered} \tag{2.11}$$

When $\lambda_k = 0$, $\nu_k = 0$, it is impossible to determine $s_{2k-1,2k-1}$ from (2.11) and in this case, therefore, to satisfy Equation (2.9) it suffices to take

$$s_{2k-1,2k-1} = C\frac{(-1)^p\lambda_{k+2}\ldots\lambda_n(\omega_k^2 - \lambda_{k+1})}{\omega_k\ldots\omega_{n-1}}$$

From (2.10) it follows that matrix S has the form

$$S = \begin{Vmatrix} 1 \dots 0 & 0 & 0 & 0 & 0 & \dots 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 \dots 1 & 0 & 0 & 0 & 0 & \dots 0 & 0 \\ 0 \dots 0 & s_{2k-1,2k-1} & 0 & 0 & s_{2k-1,2k+2} & \dots 0 & s_{2k-1,2n} \\ 0 \dots 0 & 0 & s_{2k,2k} & s_{2k,2k+1} & 0 & \dots s_{2k,2n-1} & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 \dots 0 & s_{2n-1,2k-1} & 0 & 0 & s_{2n-1,2k+2} & \dots 0 & s_{2n-1,2n} \\ 0 \dots 0 & 0 & 0 & s_{2n,2k} & s_{2n,2k+1} & \dots s_{2n,2n-1} & 0 \end{Vmatrix}$$

Hence it follows that

$$|S| = \begin{vmatrix} s_{2k-1,2k-1} & s_{2k-1,2k+2} & \dots & s_{2k-1,2n} \\ s_{2k+1,2k-1} & s_{2k+1,2k+2} & \dots & s_{2k+1,2n} \\ \dots & \dots & \dots & \dots \\ s_{2n-1,2k-1} & s_{2n-1,2k+2} & \dots & s_{2n-1,2n} \end{vmatrix} \cdot \begin{vmatrix} s_{2k,2k} & s_{2k,2k+1} & \dots & s_{2k,2n-1} \\ s_{2k+2,2k} & s_{2k+2,2k+1} & \dots & s_{2k+2,2n-1} \\ \dots & \dots & \dots & \dots \\ s_{2n,2k} & s_{2n,2k+1} & \dots & s_{2n,2n-1} \end{vmatrix} =$$

$$= (-1)^{\alpha} \frac{\lambda_k \dots \lambda_n \lambda_{k+1} \lambda^2_{k+2} \dots \lambda_n^{p-1}}{\omega_k^2 \omega_{k+1}^4 \dots \omega_{n-1}^{2(p-1)} \nu_k \dots \nu_n} (\nu_{k+1} - \nu_k)^2 \dots (\nu_n - \nu_{n-1})^2 \tag{2.12}$$

when $\lambda_k \neq 0,\ \nu_k \neq 0$, and when $\lambda_k = \nu_k = 0$

$$|S| = c(-1)^{\alpha} \frac{\lambda_{k+1} \dots \lambda_n \lambda_{k+2} \lambda^2_{k+3} \dots \lambda_n^{p-2}}{\omega_k \omega_{k+1}^3 \dots \omega_{n-1}^{2p-3}} (\nu_{k+2} - \nu_{k+1})^2 \dots (\nu_n - \nu_{n-1})^2 \nu_{k+1} \dots \nu_n$$

From (2.12) it follows that $|S| \neq 0$ for any λ_i satisfying conditions (1.5) (in (2.12) c can always be presupposed nonzero since it suffices to assume $\omega_k^2 \neq \lambda_{k+1}$, and this is always possible).

S e c o n d c a s e . Let $p = n - k + 1$ be an odd number. Then under (2.10) one of the solutions of system (2.9) will be

for $\lambda_k \neq 0\ (\nu_k \neq 0)$ (2.13)

$$s_{2i-1,2k+4j-5} = \lambda_{k+2j-2} \frac{\Delta_{2p-4j+2}(\nu_i)}{\omega_{n-1} \dots \omega_{k+2j-4}} \nu_i^{j-2} \qquad \left(j = 1, \dots, \frac{p-1}{2},\ i = k, \dots, n\right)$$

$$s_{2i,2k+4j-4} = \frac{\Delta_{2p-4j+2}(\nu_i)}{\omega_{n-1} \dots \omega_{k+2j-2}} \nu_i^{j-1} \qquad \left(j = 1, \dots, \frac{p+1}{2},\ i = k, \dots, n\right)$$

$$s_{2i-1,2k+4j-2} = \frac{\Delta_{2p-4j}(\nu_i)}{\omega_{n-1} \dots \omega_{k+2j-1}} \nu_i^{j-1} \qquad \left(j = 1, \dots, \frac{p-1}{2},\ i = k, \dots, n\right)$$

$$s_{2i,2k+4j-3} = \lambda_{k+2j-1} \frac{\Delta_{2p-4j}(\nu_i)\, \nu_i^{j-1}}{\omega_{n-1} \dots \omega_{k+2j-1}} \qquad \left(j = 1, \dots, \frac{p-1}{2},\ i = k, \dots, n\right)$$

for $\lambda_k = 0\ (\nu_k = 0)$

$$s_{2k-1,2k-1} = (-1)^p \frac{(\omega_k^2 - \lambda_{k+1}) \lambda_{k+2} \dots \lambda_n}{\omega_{n-1} \dots \omega_k} = c$$

Computing the determinant $|S|$ in precisely the same way as for an even p, we get for $\lambda_k \neq 0$ $(\nu_k \neq 0)$

$$|S| = (-1)^\alpha \frac{\lambda_k \ldots \lambda_n \lambda_{k+1} \ldots \lambda_n^{p-1}}{\omega_k^2 \ldots \omega_{n-1}^{2p-2} \nu_k \ldots \nu_n} (\nu_{k+1} - \nu_k)^2 \ldots (\nu_n - \nu_{n-1})^2$$

for $\lambda_k = 0$ $(\nu_k = 0)$

$$|S| = c(-1)^\alpha \frac{\lambda_{k+1} \ldots \lambda_n \lambda_{k+1} \ldots \lambda_n^{p-2}}{\omega_k \omega_{k+1}^3 \ldots \omega_{n-1}^{2p-3}} \nu_{k+1} \ldots \nu_n (\nu_{k+2} - \nu_{k+1})^2 \ldots (\nu_n - \nu_{n-1})^2$$

In this case also $|S| \neq 0$ under conditions (1.5).

Let us, however, ascertain whether system (2.1) is completely controllable by the controlling action u. For this we write out the $2n$-dimensional vector, the column Sb. From (1.4),(1.5) and (2.10) follows:

$$Sb = \{0, \alpha_1, \ldots, 0, \alpha_k S_{2k2k}, \ldots, 0, \alpha_k, s_{2n,2k}\} \qquad (2.14)$$

But since according to (2.8) the matrix $L = SAS^{-1}$ has a quasidiagonal structure precisely of the same kind as treated in [3] and since the vector Sb coincides with the vector b, the conditions for the complete controllability of system (2.7) will be

$$\begin{array}{llll} \alpha_i \neq 0, \quad \lambda_i \neq \lambda_j & (i, j = 1, \ldots, k-1), & \alpha_k s_{2i2k} \neq 0 & (i = k, \ldots, n) \\ \lambda_i \neq \nu_j & (i = 1, \ldots, k-1, j = k, \ldots, n), & \nu_i \neq \nu_j & (i, j = k, \ldots, n) \end{array} \qquad (2.15)$$

But in accordance with conditions (1.5) and with the proof carried out above, conditions (2.15) are fulfilled, i.e. system (2.7) and, consequently, also (2.1) are completely controllable under a suitable choice of $\omega_k, \ldots, \omega_{n-1}$.

It should be noted that the gyroscopic forces may contribute to the improvement of the observability of the system, namely, is system (1.4), not being completely observable with respect to the coordinate $\xi = \Sigma\alpha_i x_{2i-1}$ is subjected to the action of gyroscopic forces, then under (1.5) we can select these forces in such a way that in the presence of these forces, system (1.4) becomes completely observable with respect to the quantity ξ. When $\lambda_* \neq 0$ and under (1.5) system (1.4), not being completely observable with respect to the rate $\xi' = (b^* x)$, can be made completely observable by applying to the system gyroscopic forces chosen in the manner indicated above.

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